

LIPSCHITZ CONTINUITY AND CONVEXITY PRESERVING FOR SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS IN THE HEISENBERG GROUP

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ABSTRACT. In this paper we study viscosity solutions of semilinear parabolic equations in the Heisenberg group. We show uniqueness of viscosity solutions with exponential growth at space infinity. We also study Lipschitz and horizontal convexity preserving properties under appropriate assumptions. Counterexamples show that in general such properties that are well-known for semilinear and fully nonlinear parabolic equations in the Euclidean spaces do not hold in the Heisenberg group.

1. INTRODUCTION

This paper is concerned with the uniqueness and the Lipschitz and convexity preserving properties for viscous Hamilton-Jacobi equations on the Heisenberg group \mathbb{H} :

$$\begin{cases} u_t - \operatorname{tr}(A(\nabla_H^2 u)^*) + f(p, \nabla_H u) = 0 & \text{in } \mathbb{H} \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{H}, \end{cases} \quad (1.1)$$

where A is a given 2×2 symmetric positive-semidefinite matrix and the function $f : \mathbb{H} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies certain assumptions to be made explicit later. Here $\nabla_H u$, $(\nabla_H^2 u)^*$ are respectively the horizontal gradient and the horizontal symmetrized Hessian of the unknown function u in space, and u_0 is a given locally uniformly continuous function in \mathbb{H} .

Many of our results in this work also hold for more general fully nonlinear degenerate parabolic equations of the type

$$u_t + F(p, \nabla_H u, (\nabla_H^2 u)^*) = 0 \quad \text{in } \mathbb{H} \times (0, \infty) \quad (1.3)$$

under proper regularity assumptions on F . We however focus on (1.1) for simplicity of exposition.

1.1. Uniqueness for unbounded solutions. Motivated by the uniqueness results in \mathbb{R}^n [10, 3], we first give a uniqueness result for unbounded viscosity solutions of (1.1)–(1.2), which is useful in our later discussion about the Lipschitz and convexity preserving properties. To this end, we need the following Lipschitz continuity of f .

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(A1) There exists $L_1 > 0$ such that

$$|f(p, w_1) - f(p, w_2)| \leq L_1 |w_1 - w_2| \quad (1.4)$$

for all $p \in \mathbb{H}$ and $w_1, w_2 \in \mathbb{R}^2$.

(A2) There exists $L_2(\rho) > 0$ depending on $\rho > 0$ such that

$$|f(p, w) - f(q, w)| \leq L_2(\rho) |p \cdot q^{-1}|_G \quad (1.5)$$

for all $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$ and all $w \in \mathbb{R}^2$.

Here $|\cdot|_G$ denotes the Korányi gauge in \mathbb{H} , i.e.,

$$|p|_G = ((x_p^2 + y_p^2)^2 + 16z_p^2)^{\frac{1}{4}}$$

for all $p = (x_p, y_p, z_p) \in \mathbb{H}$. Note that (A2) is not the usual local Lipschitz continuity in \mathbb{H} , since the distance between $p, q \in \mathbb{H}$ defined by $d_R(p, q) = |p \cdot q^{-1}|_G$ is invariant only under right translations and therefore not equivalent to the usual gauge metric give by $d_L(p, q) = |p^{-1} \cdot q|_G$ or the Carnot-Carathéodory metric; see Section 2.2 for more details.

Our comparison principle is as below.

Theorem 1.1 (Comparison principle for unbounded solutions). *Assume that the Lipschitz conditions (A1) and (A2) hold. Let u and v be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of (1.1). Assume that for any fixed $T > 0$, there exist $k > 0$ and $C_T > 0$ depending on T such that*

$$u(p, t) - v(p, t) \leq C_T e^{k\langle p \rangle} \quad (1.6)$$

for all $(p, t) \in \mathbb{H} \times [0, T]$, where

$$\langle p \rangle = (1 + x^4 + y^4 + 16z^2)^{\frac{1}{4}} \quad \text{for all } p = (x, y, z) \in \mathbb{H}. \quad (1.7)$$

If $u(p, 0) \leq v(p, 0)$ for all $p \in \mathbb{H}$, then $u \leq v$ in $\mathbb{H} \times [0, \infty)$.

As an immediate consequence (Corollary 3.1), viscosity solutions of (1.1) are unique within the class of functions satisfying the following exponential growth condition at infinity:

(G) For any $T > 0$, there exists $k > 0$ and $C_T > 0$ such that $|u(p, t)| \leq C_T e^{k\langle p \rangle}$ for all $(p, t) \in \mathbb{H} \times [0, T]$.

Uniqueness of viscosity solutions of various nonlinear equations in the Heisenberg group are studied in [5, 26, 6, 30, 24] etc. It turns out that one may extend the Euclidean viscosity theory (e.g., [9]) to sub-Riemannian manifolds. But most of these results are either for a bounded domain or for bounded solutions. It is less understood when the domain and solution are both unbounded in the Heisenberg group. To the best of our knowledge, the only known result on uniqueness for time-dependent equations in this case is due to Haller Martin [15], where a comparison principle is established for a class of nonlinear parabolic equations including the horizontal Gauss curvature flow of graphs in the Carnot group. The comparison principle in [15] is for solutions with polynomial growth at infinity while ours is for exponential growth, but our assumptions on the structure of the equations are stronger.

1.2. Lipschitz and convexity preserving. In the Euclidean space, Lipschitz continuity and convexity preserving are two very important properties, closely related to the maximum or comparison principle, which hold for a large class of linear and nonlinear parabolic equations: when the initial value u_0 is Lipschitz continuous (resp., convex), the unique solution $u(x, t)$ is Lipschitz continuous (resp., convex) in x as well for any $t \geq 0$. Concerning the convexity preserving property in \mathbb{R}^n , we refer the reader to [21, 19, 28, 14, 1, 13, 17] for a standard PDE approach in different contexts based on convexity (or concavity) maximum principles and [22] for proofs using the discrete games introduced in [20, 27, 25].

In what follows, assuming appropriate growth conditions for the initial value u_0 and its derivatives, we sketch a proof of these properties for the unique smooth solution of the classical heat equation:

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \quad (1.8)$$

with $u(\cdot, 0) = u_0(\cdot)$ in \mathbb{R}^n , where Δu denotes the usual (Euclidean) Laplacian operator acted on u .

By differentiating the equation with respect to the space variables, one may easily see that each of the components of ∇u satisfies the heat equation (1.8), which, by the maximum principle, implies that $\nabla u(\cdot, t)$ is bounded for any $t \geq 0$ if ∇u_0 is bounded in \mathbb{R}^n .

A similar argument works for the convexity preserving property. Indeed, it is not difficult to find that, for any fixed vector $w \in \mathbb{R}^n$, $\langle \nabla^2 u w, w \rangle$ satisfies the heat equation. One may apply the maximum principle again to show $\langle \nabla^2 u(\cdot, t) w, w \rangle \geq 0$ for any $t \geq 0$ if it holds initially, which is equivalent to the statement of convexity preserving.

We intend to extend these preserving properties to nonlinear equations in the Heisenberg group \mathbb{H} . Notions and properties of Lipschitz continuity and convexity in the Heisenberg group are available in the literature [11, 23, 18]. In fact, a function u is said to be Lipschitz continuous in \mathbb{H} if there exists $L > 0$ such that

$$|u(p) - u(q)| \leq L d_L(p, q)$$

for all $p, q \in \mathbb{H}$, and u is said to be horizontally convex in \mathbb{H} if

$$u(p \cdot h^{-1}) + u(p \cdot h) \geq 2u(p)$$

for any $p \in \mathbb{H}$ and any $h \in \mathbb{H}_0$, where

$$\mathbb{H}_0 = \{h \in \mathbb{H} : h = (h_1, h_2, 0) \text{ for } h_1, h_2 \in \mathbb{R}\}.$$

It is clear that Lipschitz continuity and horizontal convexity are both left invariant.

It is worth stressing that our generalization is by no means immediate. As observed above, besides necessary applications of a comparison principle, the key in the straightforward proofs for the Euclidean case lies at differentiating the equation and interchanging derivatives. This is however not applicable directly in the Heisenberg group, since the mixed second derivatives in the Heisenberg group are not commutative in general. In fact, our counterexamples show that preserving of

Lipschitz continuity and horizontal convexity may fail even for very simple linear equations; see Examples 4.1 and 5.1 for the linear equation

$$u_t - \langle h_0, \nabla_H u \rangle = 0 \quad \text{in } \mathbb{H}, \quad (1.9)$$

where $h_0 \in \mathbb{R}^2$ is given. Its unique viscosity solution turns out to be right translations of the initial value.

Since the horizontal gradient $\nabla_H u$ and horizontal Hessian $\nabla_H^2 u$ are not in general right invariant but only left invariant, we cannot rely on the symmetry of second derivatives for our study of Lipschitz and convexity preserving properties.

On the other hand, there are many examples on Lipschitz and convexity preserving in the Heisenberg group. One sufficient condition for the equivalence between Lipschitz continuity/horizontal convexity of a function with respect to both metrics d_L and d_R is evenness or vertical evenness of the function; see Definition 2.4, Proposition 2.5 and Proposition 5.6. Another sufficient condition for the equivalence of both convexity notions is a separable structure of the function (Proposition 5.7).

We thus can obtain the Lipschitz continuity and convexity preserving properties by first investigating them with respect to the right invariant metric d_R and then using the additional assumptions above. Let us present our results in a simpler case.

Theorem 1.2 (Preserving of right invariant Lipschitz continuity). *Assume that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is Lipschitz. Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of*

$$u_t - \text{tr}(A(\nabla_H^2 u)^*) + f(\nabla_H u) = 0 \quad \text{in } \mathbb{H} \times (0, \infty), \quad (1.10)$$

with $u(\cdot, 0) = u_0(\cdot)$ satisfying the growth condition (G). If there exists $L > 0$ such that

$$|u_0(p) - u_0(q)| \leq L d_R(p, q)$$

for all $p, q \in \mathbb{H}$, then

$$|u(p, t) - u(q, t)| \leq L d_R(p, q)$$

for all $p, q \in \mathbb{H}$ and $t \geq 0$.

Theorem 1.2 is a direct application of Theorem 1.1. A more general version is given below in Theorem 4.2. It implies the Lipschitz preserving property of an even function or vertically even function (Corollary 4.3).

For the case of first order Hamilton-Jacobi equations ($A = 0$), if in addition we assume that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is in the form that $f(\xi) = m(|\xi|)$ with $m : \mathbb{R} \rightarrow \mathbb{R}$ locally uniformly continuous, then the Lipschitz preserving property of a bounded solution can be directly shown without the evenness assumption. We refer the reader to Theorem 4.4, which answers a question asked in [26]. A more general question on Lipschitz continuity of viscosity solutions was posed in [2], but it is not clear if our method here immediately applies to that general setting.

As for the h-convexity preserving property, we obtain the following.

Theorem 1.3 (Right invariant h-convexity preserving). *Assume that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is Lipschitz. Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of (1.10) with $u(\cdot, 0) = u_0(\cdot)$*

satisfying the growth condition (G). Assume in addition that f is concave in \mathbb{R}^2 , i.e.,

$$f(\xi) + f(\eta) \leq 2f\left(\frac{1}{2}(\xi + \eta)\right) \quad (1.11)$$

for all $\xi, \eta \in \mathbb{R}^2$. If u_0 is right invariant h -convex in \mathbb{H} ; that is,

$$u_0(h^{-1} \cdot p) + u_0(h \cdot p) \geq 2u(p)$$

for all $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$, then so is $u(\cdot, t)$ for all $t \geq 0$.

The convexity preserving property for solutions that are either even or in a separable form follows easily (Corollary 5.8).

Our study of the convexity preserving property in the Heisenberg group is also inspired by recent works on horizontal mean curvature flow in sub-Riemannian manifolds [7, 12]. The mean curvature flow in \mathbb{R}^n is known to preserve convexity [16], but it is not clear if such a property also holds in \mathbb{H} in general. Our analysis about convexity is only for the simpler equation (1.1). However, an explicit solution of the mean curvature flow in \mathbb{H} that does preserve convexity can be found in Example 5.11; see also [12].

In the proof of Theorem 1.3, we show a convexity maximum principle, following the proof of Theorem 1.1. A general version of this theorem for the equation (1.1) is given in Theorem 5.3, where $f : \mathbb{H} \times \mathbb{R}^2$ is assumed to be (right invariant) concave. One may further generalize this result for (1.3) by assuming that F is concave in all arguments. We remark that in the Euclidean case as studied in [14, 17] etc., there is no need to assume (1.11). We need this assumption due to lack of an equivalent definition of horizontal convexity in terms of averages of endpoints. More precisely, convexity of a function $u \in C(\mathbb{R}^n)$ can be expressed by

$$u(\xi) + u(\eta) \geq 2u\left(\frac{\xi + \eta}{2}\right)$$

for any $\xi, \eta \in \mathbb{R}^n$. However, for horizontal convexity in \mathbb{H} there is no such “global” expression valid for all pairs $p, q \in \mathbb{H}$ that are only horizontally related, i.e., $p = q \cdot h$ for $h \in \mathbb{H}_0$. It is not clear to us if this assumption can be dropped in our theorem.

This paper is organized in the following way. In Section 2, we present some basic and useful facts about the Heisenberg group, including an introduction of its metrics, Lipschitz continuity and horizontal convexity. In Section 3, we give a proof of Theorem 1.1 and also include an existence result at the end. The Lipschitz preserving property is studied in Section 4. Section 5 is dedicated to a discussion of convexity preserving property with several explicit examples in Section 5.3.

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2. PRELIMINARIES

2.1. Review of the Heisenberg group \mathbb{H} . Recall that the Heisenberg group \mathbb{H} is \mathbb{R}^3 endowed with the non-commutative group multiplication

$$(x_p, y_p, z_p) \cdot (x_q, y_q, z_q) = \left(x_p + x_q, y_p + y_q, z_p + z_q + \frac{1}{2}(x_p y_q - x_q y_p) \right),$$

for all $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$ in \mathcal{H} . Note that the group inverse of $p = (x_q, y_q, z_q)$ is $p^{-1} = (-x_q, -y_q, -z_q)$. The Korányi gauge is given by

$$|p|_G = ((p_1^2 + p_2^2)^2 + 16p_3^2)^{1/4},$$

and the left-invariant Korányi or gauge metric is

$$d_L(p, q) = |p^{-1} \cdot q|_G.$$

The Lie Algebra of \mathbb{H} is generated by the left-invariant vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}; \\ X_2 &= \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}; \\ X_3 &= \frac{\partial}{\partial z}. \end{aligned}$$

One may easily verify the commuting relation $X_3 = [X_1, X_2] = X_1 X_2 - X_2 X_1$.

The horizontal gradient of u is given by

$$\nabla_H u = (X_1 u, X_2 u)$$

and the symmetrized second horizontal Hessian $(\nabla_H^2 u)^* \in S^{2 \times 2}$ is given by

$$(\nabla_H^2 u)^* := \begin{pmatrix} X_1^2 u & (X_1 X_2 u + X_2 X_1 u)/2 \\ (X_1 X_2 u + X_2 X_1 u)/2 & X_2^2 u \end{pmatrix}.$$

Here $S^{n \times n}$ denotes the set of all $n \times n$ symmetric matrices.

A piecewise smooth curve $s \mapsto \gamma(s) \in \mathbb{H}$ is called horizontal if its tangent vector $\gamma'(s)$ is in the linear span of $\{X_1(\gamma(s)), X_2(\gamma(s))\}$ for every s such that $\gamma'(s)$ exists; in other words, there exist $a(s), b(s) \in \mathbb{R}$ satisfying

$$\gamma'(s) = a(s)X_1(\gamma(s)) + b(s)X_2(\gamma(s))$$

whenever $\gamma'(s)$ exists. We denote

$$\|\gamma'(s)\| = (a^2(s) + b^2(s))^{\frac{1}{2}}.$$

Given $p, q \in \mathbb{H}$, denote

$$\Gamma(p, q) = \{\text{horizontal curves } \gamma(s) \ (s \in [0, 1]): \gamma(0) = p \text{ and } \gamma(1) = q\}.$$

Chow's theorem states that $\Gamma(p, q) \neq \emptyset$; see, for example, [4]. The Carnot-Carathéodory metric is then defined to be

$$d_{CC}(p, q) = \inf_{\gamma \in \Gamma(p, q)} \int_0^1 \|\gamma'(s)\| ds.$$

2.2. Metrics on \mathbb{H} . Besides the left-invariant Korányi metric d_L and Carnot-Carathéodory metric d_{CC} , the function $d_R(p, q) = |p \cdot q^{-1}|_G$ for any $p, q \in \mathbb{H}$ defines another metric on H , which is right invariant; in fact, $d_R(p, q) = d_L(p^{-1}, q^{-1})$ for any $p, q \in \mathbb{H}$.

It is known that d_L is bi-Lipschitz equivalent to the Carnot-Carathéodory metric d_{CC} [8, 24]. The metrics d_L and d_R are not bi-Lipschitz equivalent, which is indicated in the example below.

Example 2.1. One may choose

$$p = (1 - \varepsilon, 1 + \varepsilon, \varepsilon), \quad q = (1, 1, 0)$$

with $\varepsilon > 0$ small, then by direct calculation, we have $d_L(p, q)^4 = |q^{-1} \cdot p|_G^4 = 4\varepsilon^4$ and $d_R(p, q)^4 = |p \cdot q^{-1}|_G^4 = 4\varepsilon^4 + 64\varepsilon^2$, which indicates that one cannot expect the existence of a constant $C > 0$ such that $d_R(p, q) \leq C d_L(p, q)$ for all $p, q \in \mathbb{H}$. A variant of this example shows that the reverse inequality also fails in general.

Although the metrics above are not bi-Lipschitz equivalent, it turns out that one is locally Hölder continuous in the other.

Proposition 2.2. *For any $\rho > 0$, there exists $C_\rho > 0$ such that*

$$d_L(p, q) \leq C_\rho d_R(p, q)^{\frac{1}{2}} \tag{2.1}$$

and

$$d_R(p, q) \leq C_\rho d_L(p, q)^{\frac{1}{2}} \tag{2.2}$$

for any $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$.

Proof. We give a proof for the sake of completeness. We only show (2.1). The proof of (2.2) is similar. Set $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$. It is then clear that we only need to show that there exists some $C > 0$ depending only on ρ such that

$$\begin{aligned} & \left| z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p \right| \\ & \leq C \left((|x_p - x_q|^2 + |y_p - y_q|^2)^2 + 16 \left(z_p - z_q - \frac{1}{2}x_p y_q + \frac{1}{2}x_q y_p \right)^2 \right)^{\frac{1}{4}} \end{aligned}$$

for all $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$.

Let $\delta = ((|x_p - x_q|^2 + |y_p - y_q|^2)^2 + 16|z_p - z_q - \frac{1}{2}x_p y_q + \frac{1}{2}x_q y_p|^2)^{\frac{1}{4}} \leq 1$. Then it is clear that

$$\begin{aligned} \left| z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p \right| & \leq \frac{\delta^2}{4} + |x_p y_q - x_q y_p| \\ & = \frac{\delta^2}{4} + |(x_p - x_q)y_q - x_q(y_p - y_q)|. \end{aligned}$$

It follows that

$$\left| z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p \right| \leq \frac{\delta^2}{4} + (|x_p - x_q|^2 + |y_p - y_q|^2)^{\frac{1}{2}}(x_q^2 + y_q^2)^{\frac{1}{2}}.$$

Noticing that $x_q^2 + y_q^2 \leq \rho^2$, we have

$$\left| z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p \right| \leq \frac{\delta^2}{4} + \rho(|x_p - x_q|^2 + |y_p - y_q|^2)^{\frac{1}{2}} \leq \frac{\delta^2}{4} + \rho\delta = \left(\frac{\delta}{4} + \rho \right) \delta.$$

We conclude the proof by choosing $C = 1/4 + \rho$. \square

2.3. Lipschitz continuity. We discuss two types of Lipschitz continuity with respect to d_L and d_R .

It is easily seen that the function $f_0 : \mathbb{H} \rightarrow \mathbb{R}$ given by $f_0(p) = |p|_G$ is a Lipschitz function with respect to d_L and d_R , due to the triangle inequality. But there exist functions that are Lipschitz with respect to one of the metrics but not with respect to the other. An example, following Example 2.1, is as below.

Example 2.3. Fix $q = (1, 1, 0) \in \mathbb{H}$ as in Example 2.1. Let $f_q : \mathbb{H} \rightarrow \mathbb{R}$ defined by $f_q(p) = d_R(p, q)$ for every $p \in \mathbb{H}$, which satisfies

$$|f_q(p) - f_q(p')| = |d_R(p, q) - d_R(p', q)| \leq d_R(p, p')$$

for all $p, p' \in \mathbb{H}$. But there is no constant $L > 0$ such that

$$f_q(p) - f_q(p') \leq L d_L(p, p')$$

for all $p, p' \in \mathbb{H}$, for otherwise we may take $p = (1 - \varepsilon, 1 + \varepsilon, \varepsilon)$ and $p' = q$, and get

$$d_R(p, p') \leq L d_L(p, p'),$$

which is not true when $\varepsilon > 0$ small, as explained in Example 2.1. However, by Proposition 2.2, the function f_q is still locally 1/2-Hölder continuous with respect to d_L .

On the other hand, not all functions that are (locally) Lipschitz with respect to d_L or d_R are (locally) Lipschitz with respect to the Euclidean metric. The simplest example is the function $f(p) = |p|_G$ for $p \in \mathbb{H}$.

We conclude this section by showing the equivalence of Lipschitz continuity with respect to both metrics for functions with symmetry. We include in our discussions two different types of evenness.

Definition 2.4 (Even functions). We say a function $f : \mathbb{H} \rightarrow \mathbb{R}$ is even (or symmetric about the origin) if $f(p) = f(p^{-1})$ for all $p \in \mathbb{H}$. We say f is vertically even (or symmetric about the horizontal coordinate plane) if $f(p) = f(\bar{p})$ for all $p \in \mathbb{H}$, where

$$\bar{p} = (x, y, -z) \text{ for any } p = (x, y, z) \in \mathbb{H}. \quad (2.3)$$

Since $|p \cdot q^{-1}|_G = |\bar{p}^{-1} \cdot \bar{q}|_G = |(p^{-1})^{-1} \cdot q^{-1}|_G$ for any $p, q \in \mathbb{H}$, the following result is obvious.

Proposition 2.5 (Equivalence of Lipschitz continuities). *Let $f : \mathbb{H} \rightarrow \mathbb{R}$ be a function that is either even or vertically even in \mathbb{H} . Then f is Lipschitz continuous with respect to d_L if and only if f is Lipschitz continuous with respect to d_R .*

2.4. Horizontal convexity.

Definition 2.6 ([23, Definition 4.1]). Let Ω be an open set in \mathbb{H} and $u : \Omega \rightarrow \mathbb{R}$ be an upper semicontinuous function. The function u is said to be horizontally convex or h-convex in Ω , if for every $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$ such that $[p \cdot h^{-1}, p \cdot h] \subset \Omega$, we have

$$u(p \cdot h^{-1}) + u(p \cdot h) \geq 2u(p). \quad (2.4)$$

One may also define convexity of a function through its second derivatives in the viscosity sense.

Definition 2.7. Let Ω be an open set in \mathbb{H} and $u : \Omega \rightarrow \mathbb{R}$ be an upper semicontinuous function. The function u is said to be v-convex in Ω if

$$(\nabla_H^2 u)^*(p) \geq 0 \quad \text{for all } p \in \mathbb{H} \quad (2.5)$$

in the viscosity sense.

It is clear that $u \in C^2(\Omega)$ is v-convex if it satisfies (2.5) everywhere in Ω . It is known that the h-convexity and v-convexity are equivalent [23, 29]. The following example shows that h-convexity is very different from convexity in the Euclidean sense.

Example 2.8. Let

$$f(x, y, z) = x^2 y^2 + 2z^2 \quad (2.6)$$

for all $(x, y, z) \in \mathbb{H}$. It is not difficult to verify that f is h-convex. Indeed, for any $p = (x, y, z) \in \mathbb{H}$ and $h = (h_1, h_2, 0) \in \mathbb{H}_0$, we have

$$\begin{aligned} & f(p \cdot h) + f(p \cdot h^{-1}) \\ &= 2x^2 y^2 + 4z^2 + 3x^2 h_2^2 + 3y^2 h_1^2 + 2h_1^2 h_2^2 + 6xyh_1 h_2 \\ &\geq 2f(p) + 3(xh_2 + yh_1)^2 + 2h_1^2 h_2^2 \geq 2f(p). \end{aligned}$$

The function f is an example of (globally) h-convex functions in \mathbb{H} that is not convex in \mathbb{R}^3 .

3. UNIQUENESS OF UNBOUNDED SOLUTIONS

In this section, motivated by a Euclidean argument in [3], we present a proof of Theorem 1.1 on a comparison principle for (1.1) with exponential growth at space infinity. Our result and proof are different from those of [15].

Proof of Theorem 1.1. We aim to show that $u \leq v$ in $\mathbb{H} \times [0, T)$ for any fixed $T > 0$. By the growth assumption, there exist $k > 0$ and $C_T > 0$ satisfying (1.6). Take an arbitrary constant $\beta > \min\{k, 1\}$ and then $\alpha > 0$ to be determined later. Set

$$g(p, t) = e^{\alpha t + \beta \langle p \rangle} \quad (3.1)$$

for $(p, t) \in \mathbb{H} \times [0, \infty)$. Recall that $\langle p \rangle$ is a function of $p \in \mathbb{H}$ given in (1.7). If $p = (x, y, z)$, we have by direct calculations

$$\nabla_H \langle p \rangle = \left(\frac{x^3 - 4yz}{(1 + x^4 + y^4 + 16z^2)^{\frac{3}{4}}}, \frac{y^3 + 4xz}{(1 + x^4 + y^4 + 16z^2)^{\frac{3}{4}}} \right), \quad (3.2)$$

which implies that there exists $\mu > 0$ such that

$$|\nabla_H g(p, t)| \leq \beta \mu g(p, t) \quad (3.3)$$

for all $(p, t) \in \mathbb{H} \times [0, \infty)$.

We assume by contradiction that $u(p, t) - v(p, t)$ takes a positive value at some $(p, t) \in \mathbb{H} \times (0, \infty)$. Then there exists $\sigma \in (0, 1)$ such that

$$u(p, t) - v(p, t) - 2\sigma g(p, t) - \frac{\sigma}{T - t}$$

attains a positive maximum at $(\hat{p}, \hat{t}) \in \mathbb{H} \times [0, T]$. For all $\varepsilon > 0$ small, consider the function

$$\Phi(p, q, t, s) = u(p, t) - v(q, s) - \sigma \Psi_\varepsilon(p, q, t, s) - \frac{(t - s)^2}{\varepsilon} - \frac{\sigma}{T - t}$$

with

$$\Psi_\varepsilon(p, q, t, s) = \varphi_\varepsilon(p, q) + K(p, q, t, s),$$

$$\varphi_\varepsilon(p, q) = \frac{1}{\varepsilon} d_R(p, q)^4 = \frac{|p \cdot q^{-1}|^4}{\varepsilon}, \quad K(p, q, t, s) = g(p, t) + g(q, s).$$

Then Φ attains a positive maximum at some $(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathbb{H}^2 \times [0, T]^2$. In particular,

$$\Phi(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \geq \Phi(\hat{p}, \hat{p}, \hat{t}, \hat{t}),$$

which implies that

$$\begin{aligned} \frac{|p_\varepsilon \cdot q_\varepsilon^{-1}|^4}{\varepsilon} + \frac{(t_\varepsilon - s_\varepsilon)^2}{\varepsilon} &\leq u(p_\varepsilon, t_\varepsilon) - v(q_\varepsilon, s_\varepsilon) - \sigma g(p_\varepsilon, t_\varepsilon) - \sigma g(q_\varepsilon, s_\varepsilon) - \frac{\sigma}{T - t_\varepsilon} \\ &\quad - \left(u(\hat{p}, \hat{t}) - v(\hat{p}, \hat{t}) - 2\sigma g(\hat{p}, \hat{t}) - \frac{\sigma}{T - \hat{t}} \right). \end{aligned} \quad (3.4)$$

Since, due to (1.6), the terms $u(p_\varepsilon, t_\varepsilon) - v(q_\varepsilon, s_\varepsilon) - \sigma g(p_\varepsilon, t_\varepsilon) - \sigma g(q_\varepsilon, s_\varepsilon)$ are bounded from above uniformly in ε , we have

$$d_R(p_\varepsilon, q_\varepsilon) \rightarrow 0 \text{ and } t_\varepsilon - s_\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (3.5)$$

We notice that $p_\varepsilon, q_\varepsilon$ are bounded, since otherwise the right hand side of (3.4) will tend to $-\infty$. Therefore, by taking a subsequence, still indexed by ε , we have $p_\varepsilon, q_\varepsilon \rightarrow \bar{p} \in \mathbb{H}$ and $t_\varepsilon, s_\varepsilon \rightarrow \bar{t} \in [0, T]$. It follows that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} u(p_\varepsilon, t_\varepsilon) - v(q_\varepsilon, s_\varepsilon) - \sigma g(p_\varepsilon, t_\varepsilon) - \sigma g(q_\varepsilon, s_\varepsilon) - \frac{\sigma}{T - t_\varepsilon} \\ \leq u(\bar{p}, \bar{t}) - v(\bar{p}, \bar{t}) - 2\sigma g(\bar{p}, \bar{t}) - \frac{\sigma}{T - \bar{t}}, \end{aligned}$$

which yields

$$\varphi_\varepsilon(p_\varepsilon, q_\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Also, it is easily seen that $\bar{t} > 0$ and therefore $t_\varepsilon, s_\varepsilon > 0$ thanks to the condition that $u(\cdot, 0) \leq v(\cdot, 0)$ in \mathbb{H} .

In order to apply the Crandall-Ishii lemma (cf. [9]) in our current case, let us recall the definition of semijets adapted to the Heisenberg group: for any $(p, t) \in \mathbb{H} \times (0, \infty)$ and any locally bounded upper semicontinuous function u in $\mathbb{H} \times (0, \infty)$,

$$P_H^{2,+}u(p, t) = \left\{ (\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^{2 \times 2} : u(q, s) \leq u(p, t) + \tau(s - t) \right. \\ \left. + \langle \zeta, p^{-1} \cdot q \rangle + \frac{1}{2} \langle Xh, h \rangle + o(|p^{-1} \cdot q|_G^2) \right\},$$

where h denotes the horizontal projection of $p^{-1} \cdot q$. Similarly, we may define

$$P_H^{2,-}u(p, t) = \left\{ (\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^{2 \times 2} : u(q, s) \geq u(p, t) + \tau(s - t) \right. \\ \left. + \langle \zeta, p^{-1} \cdot q \rangle + \frac{1}{2} \langle Xh, h \rangle + o(|p^{-1} \cdot q|_G^2) \right\}$$

for any locally bounded lower semicontinuous function u . Also, the closure $\overline{P}_H^{2,+}$ is the set of triples $(\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^{2 \times 2}$ that satisfy the following: there exist $(p_j, t_j) \in \mathbb{H} \times [0, \infty)$ and $(\tau_j, \zeta_j, X_j) \in P_H^{2,+}(p_j, t_j)$ such that

$$(p_j, t_j, u(p_j, t_j), \tau_j, \zeta_j, X_j) \rightarrow (p, t, u(p, t), \tau, \zeta, X) \quad \text{as } j \rightarrow \infty.$$

The closure set $\overline{P}_H^{2,-}$ of $P_H^{2,-}$ can be similarly defined. We refer to [6] for more details.

We now apply the adaptation of the Crandall-Ishii lemma to the Heisenberg group [24, 6] and get for any $\lambda \in (0, 1)$

$$(a_1, \zeta_1, X) \in \overline{P}_H^{2,+}u(p_\varepsilon, t_\varepsilon) \text{ and } (a_2, \zeta_2, Y) \in \overline{P}_H^{2,-}v(q_\varepsilon, s_\varepsilon)$$

such that

$$a_1 - a_2 = \alpha \sigma K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) + \frac{\sigma}{(T - t_\varepsilon)^2}, \quad (3.6)$$

$$\langle Xw, w \rangle - \langle Yw, w \rangle \leq \langle (\sigma M + \lambda \sigma^2 M^2)w_{p_\varepsilon} \oplus w_{q_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \rangle, \quad (3.7)$$

and the horizontal projections of $\zeta_1, \zeta_2 \in \mathbb{R}^3$ can be written respectively as $\xi + \eta_1$ and $\xi + \eta_2$ (in \mathbb{R}^2) with

$$\xi = \nabla_H^p \varphi_\varepsilon(p_\varepsilon, q_\varepsilon) = -\nabla_H^q \varphi_\varepsilon(p_\varepsilon, q_\varepsilon), \\ \eta_1 = \beta \sigma \nabla_H g(p_\varepsilon, t_\varepsilon), \quad \eta_2 = -\beta \sigma \nabla_H g(q_\varepsilon, s_\varepsilon).$$

Here $w = (w_1, w_2) \in \mathbb{R}^2$ is arbitrary, $M = (\nabla^2 \Psi_\varepsilon)^*(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon)$ is a 6×6 symmetric matrix, and

$$w_{p_\varepsilon} = \left(w_1, w_2, \frac{1}{2} w_2 x_{p_\varepsilon} - \frac{1}{2} w_1 y_{p_\varepsilon} \right) \quad (3.8)$$

and

$$w_{q_\varepsilon} = \left(w_1, w_2, \frac{1}{2} w_2 x_{p_\varepsilon} - \frac{1}{2} w_1 y_{q_\varepsilon} \right) \quad (3.9)$$

with $p_\varepsilon = (x_{p_\varepsilon}, y_{p_\varepsilon}, z_{p_\varepsilon})$ and $q_\varepsilon = (x_{q_\varepsilon}, y_{q_\varepsilon}, z_{q_\varepsilon})$.

It is easily seen that $M = M_1 + M_2$, where

$$M_1 = \nabla^2 \varphi_\varepsilon(p_\varepsilon, q_\varepsilon)$$

and

$$M_2 = \nabla^2 K(p_\varepsilon, q_\varepsilon) = \begin{pmatrix} \nabla^2 g(p_\varepsilon, t_\varepsilon) & 0 \\ 0 & \nabla^2 g(q_\varepsilon, s_\varepsilon) \end{pmatrix}.$$

It follows from the calculation in the comparison arguments in [6] (and also [5, 24]) that there exists $C > 0$ such that

$$\langle (M_1 + \lambda M_2)w_{p_\varepsilon} \oplus w_{q_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \rangle \leq \frac{C}{\varepsilon} |w|^2 (z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon}y_{p_\varepsilon})^2 \quad (3.10)$$

for any $\lambda > 0$ small. We next follow the strategy in the Euclidean case from [3]. However, the algebraic complexity is quite more challenging in the non-commutative case. With the help of a computer algebra system¹, we simplify the left hand side of the following inequalities and obtain a constant $C_\beta > 0$ depending only on β , such that

$$\langle M_2(w_{p_\varepsilon} \oplus w_{q_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon}) \rangle \leq \frac{1}{\varepsilon} |w|^2 C_\beta K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon), \quad (3.11)$$

$$\begin{aligned} & \langle M_1 M_2(w_{p_\varepsilon} \oplus w_{q_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon}) \rangle \\ & \leq \frac{1}{\varepsilon} |w|^2 C_\beta K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \left| z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon}y_{p_\varepsilon} \right| \end{aligned} \quad (3.12)$$

and

$$\langle M_2^2(w_{p_\varepsilon} \oplus w_{q_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon}) \rangle \leq |w|^2 C_\beta K^2(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon). \quad (3.13)$$

We remark that the existence of C_β here is essentially due to the boundedness of $\nabla_H \langle p \rangle$ and $\nabla_H^2 \langle p \rangle$ in \mathbb{H} .

By (3.7) and (3.10), we may take $\lambda > 0$ sufficiently small, depending on the size of ε , \bar{p} , \bar{t} , and β , such that

$$\begin{aligned} & \langle Xw, w \rangle - \langle Yw, w \rangle \\ & \leq \frac{C\sigma}{\varepsilon} |w|^2 (z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon}y_{p_\varepsilon})^2 + 2\sigma |w|^2 C_\beta K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon). \end{aligned} \quad (3.14)$$

We next apply the definition of viscosity sub- and supersolutions and get

$$a_1 - \text{tr}(AX) + f(p_\varepsilon, \xi + \eta_1) \leq 0 \quad (3.15)$$

and

$$a_2 - \text{tr}(AY) + f(q_\varepsilon, \xi + \eta_2) \geq 0. \quad (3.16)$$

By subtracting (3.16) from (3.15), we have

$$a_1 - a_2 \leq \text{tr}(AX) - \text{tr}(AY) + f(q_\varepsilon, \xi + \eta_2) - f(p_\varepsilon, \xi + \eta_1),$$

which yields, by (3.14) and (A1),

$$\begin{aligned} a_1 - a_2 & \leq \frac{C\sigma}{\varepsilon} \|A\| (z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon}y_{p_\varepsilon})^2 + L_2(\rho) |p_\varepsilon \cdot q_\varepsilon^{-1}| \\ & \quad + (2\sigma C_\beta \|A\| + 2\beta \mu \sigma L_1) K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \end{aligned} \quad (3.17)$$

¹Program is available in the arXiv.org version of the paper.

with $\rho = |\bar{p}| + 1$ for $\varepsilon > 0$ sufficiently small.

Since we have (3.5), we now can take $\varepsilon > 0$ small to get

$$\frac{C}{\varepsilon} \left(z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2} x_{p_\varepsilon} y_{q_\varepsilon} + \frac{1}{2} x_{q_\varepsilon} y_{p_\varepsilon} \right)^2 + L_2(\rho) |p_\varepsilon \cdot q_\varepsilon^{-1}| \leq \frac{\sigma}{T^2}.$$

Taking $\lambda > 0$ accordingly small and

$$\alpha > 2C_\beta \|A\| + \beta \mu L_f,$$

we reach a contradiction to (3.6). □

An immediate consequence is certainly the uniqueness of solutions with at most exponential growth at space infinity.

Corollary 3.1 (Uniqueness of solutions). *Assume that (A1) and (A2) hold. Let $u_0 \in C(\mathbb{H})$. Then there is at most one continuous viscosity solution u of (1.1)–(1.2) satisfying the exponential growth condition (G).*

The existence of viscosity solutions of (1.1)–(1.2) is not the main topic of this work, but we remark that it is possible to adapt Perron's method [9] to our current case in the Heisenberg group, under various extra assumptions on the function f . For example, one may further assume on (1.1) that

$$(A3) \quad |f(p, \xi)| \leq C_f(1 + |\xi|) \text{ for some } C_f > 0 \text{ and all } p \in \mathbb{H}, \xi \in \mathbb{R}^2.$$

In this case, it is not difficult to verify by computation that $\bar{u} = Cg(p, t) + C_f t$ and $\underline{u} = -Cg(p, t) - C_f t$ are respectively a supersolution and a subsolution of (1.1) for any $C > 0$ and $\beta > 0$ when $\alpha > 0$ is sufficiently large. Indeed, we have

$$\begin{aligned} \bar{u}_t &= C\alpha g + C_f, \\ |\operatorname{tr}(A(\nabla_H^2 \bar{u})^*)| &\leq C\|A\|\beta^2 \mu^2 g, \end{aligned}$$

and

$$|f(p, \nabla_H \bar{u})| \leq CC_f \beta \mu g + C_f,$$

where μ is the same constant as in the proof of Theorem 1.1. Therefore, by (A3), we get

$$\bar{u}_t - \operatorname{tr}(A(\nabla_H^2 \bar{u})^*) + f(p, \nabla_H \bar{u}) \geq 0$$

when $\alpha > \|A\|\beta^2 \mu^2 + C_f \beta \mu$. The verification for \underline{u} is similar.

If there exist $C > 0$ and $k > 0$ such that

$$-Ce^{k\langle p \rangle} \leq u_0(p) \leq Ce^{k\langle p \rangle} \quad \text{for all } p \in \mathbb{H}, \quad (3.18)$$

then classical arguments [9] show that the supremum over all subsolutions bounded by \underline{u} and \bar{u} is in fact a unique continuous solution. We state the result below without more details in its proof.

Corollary 3.2. *Assume the Lipschitz conditions (A1), (A2) and the growth condition (A3). Let $u_0 \in C(\mathbb{H})$ satisfy (3.18) for some $C > 0$ and $k > 0$. Then there exists a unique continuous solution u of (1.1)–(1.2) satisfying the exponential growth condition (G).*

4. LIPSCHITZ PRESERVING PROPERTIES

In this section, we strengthen the assumption (A2) on f ; we assume

(A2') the function $f(p, \xi)$ is globally Lipschitz continuous in p with respect to the metric d_R , i.e., there exists $L_0 > 0$ such that

$$|f(p, \xi) - f(q, \xi)| \leq L_0 |p \cdot q^{-1}|_G \quad (4.1)$$

for all $p, q \in \mathbb{H}$ and $\xi \in \mathbb{R}^2$.

4.1. Right invariant Lipschitz continuity preserving. We first discuss the Lipschitz continuity based on the standard gauge metric d_L (or equivalently, the Carnot-Carathéodory metric). It turns out that even the simplest first order linear equation will not preserve such Lipschitz continuity.

Example 4.1. Fix $h_0 = (1, 1) \in \mathbb{R}^2$. Let us consider the equation

$$u_t - \langle h_0, \nabla_H u \rangle = 0 \quad \text{in } \mathbb{H}$$

with $u(p, 0) = u_0(p) = |p|_G$ for $p \in \mathbb{H}$. By direct verification and Corollary 3.1, the unique solution is

$$u(p, t) = |p \cdot ht|_G = d_R(p, h^{-1}t),$$

where $h = (1, 1, 0) \in \mathbb{H}_0$. However, it is not Lipschitz continuous with respect to d_L . Indeed, similar to Example 2.3, one may choose $p_1 = (-t - \varepsilon, -t + \varepsilon, -\varepsilon t)$ and $p_2 = tv^{-1} = (-t, -t, 0)$, which gives

$$u(p_1, t) - u(p_2, t) = |p_1 \cdot p_2^{-1}|_G = (4\varepsilon^4 + 64\varepsilon^2 t^2)^{\frac{1}{4}}$$

but

$$d_L(p_1, p_2) = |p_1^{-1} \cdot p_2|_G = \sqrt{2}\varepsilon.$$

The example above directs us to first consider the Lipschitz continuity with respect to d_R . The following result is an immediate consequence of Theorem 1.1.

Theorem 4.2 (Preserving of right invariant Lipschitz continuity). *Assume that $f : \mathbb{H} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the assumptions (A1), (A2') and (A3). Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of (1.1)–(1.2) satisfying the growth condition (G). If there exists $L > 0$ such that*

$$|u_0(p) - u_0(q)| \leq L d_R(p, q) \quad (4.2)$$

for all $p, q \in \mathbb{H}$, then

$$|u(p, t) - u(q, t)| \leq (L + L_0 t) d_R(p, q) \quad (4.3)$$

for all $p, q \in \mathbb{H}$ and $t \geq 0$. In particular, there exists $C_\rho > 0$ depending on $\rho > 0$ and $t \geq 0$ such that

$$|u(p, t) - u(q, t)| \leq C_\rho d_L(p, q)^{\frac{1}{2}} \quad (4.4)$$

for all $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$. Moreover, when f does not depend on the space variable p , (4.3) holds with $L_0 = 0$.

Proof. By symmetry, we only need to prove that

$$u(p, t) - u(h^{-1} \cdot p, t) \leq (L + L_0 t)|h|_G \quad (4.5)$$

for all $p, h \in \mathbb{H}$ and $t \geq 0$. It suffices to show that

$$v(p, t) = u(h^{-1} \cdot p, t) + (L + L_0 t)|h|_G$$

is a supersolution of (1.1)–(1.2) for any $h \in \mathbb{H}$. To this end, we recall the left invariance of horizontal derivatives in the Heisenberg group, which implies that v is a supersolution of

$$v_t - \text{tr}(A \nabla_H^2 v) + f(h^{-1} \cdot p, \nabla_H v) = L_0 |h|_G \quad \text{in } \mathbb{H} \times (0, \infty).$$

Since

$$|f(h^{-1} \cdot p, \nabla_H v) - f(p, \nabla_H v)| \leq L_0 |h|_G$$

due to (4.1), we easily see that v is a supersolution of (1.1). Also, by (4.2), we have $u(p, 0) \leq v(p, 0)$ for all $p \in \mathbb{H}$. We conclude the proof of (4.5) by applying Theorem 1.1. The Hölder continuity (4.4) follows from Proposition 2.2. \square

In view of Proposition 2.5, we may use the theorem above to show the preserving of Lipschitz continuity in the standard gauge metric under the assumption of evenness or vertical evenness.

Corollary 4.3 (Lipschitz preserving of even solutions). *Assume that $f : \mathbb{H} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the conditions (A1), (A2') and (A3). Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of (1.1)–(1.2) satisfying the growth condition (G). Assume also that $u(\cdot, t)$ is an even or vertically even function. If there exists $L > 0$ such that*

$$|u_0(p) - u_0(q)| \leq L d_L(p, q) \quad (4.6)$$

for all $p, q \in \mathbb{H}$, then

$$|u(p, t) - u(q, t)| \leq (L + L_0 t) d_L(p, q) \quad (4.7)$$

for all $p, q \in \mathbb{H}$ and $t \geq 0$. In particular, when f does not depend on the space variable p , then (4.7) holds with $L_0 = 0$.

4.2. A special class of Hamilton-Jacobi equations. We discuss the Lipschitz preserving property for bounded solutions of a special class of first order Hamilton-Jacobi equations whose Hamiltonians depend only on the norm of horizontal gradient. More precisely, we study equations in the form of

$$u_t + m(|\nabla_H u|) = 0 \quad \text{in } \mathbb{H} \times (0, \infty), \quad (4.8)$$

where $m : \mathbb{R} \rightarrow \mathbb{R}$ is a locally uniformly continuous function, with initial condition $u(\cdot, 0) = u_0(\cdot)$ bounded Lipschitz continuous with respect to d_L in \mathbb{H} . Since the assumption on m is quite weak, our uniqueness and existence results for unbounded solutions in Section 3 do not apply.

For solutions bounded in space, see [26] for a uniqueness theorem and a Hopf-Lax formula when the Hamiltonian $\xi \mapsto m(|\xi|)$ is assumed to be convex and coercive. For instance, when $m(|\xi|) = |\xi|^2/2$, the unique solution of (4.8) can be expressed as

$$u(p, t) = \inf_{q \in \mathbb{H}} \left\{ \frac{t}{2} d_{CC}^2 \left(0, \left(\frac{q^{-1} \cdot p}{t} \right) \right) + u_0(q) \right\}. \quad (4.9)$$

The Lipschitz preserving property (with respect to d_L or d_{CC}) was left as an open question in [26]; see also [2] for a related open question but for more general Hamiltonians. In contrast to the Euclidean case, it is not obvious how to prove the Lipschitz continuity by using the Hopf-Lax formula (4.9). We here give an answer to this question using a PDE approach.

Theorem 4.4 (Lipschitz preserving for special Hamilton-Jacobi equations). *Suppose that $m : \mathbb{R} \rightarrow \mathbb{R}$ is locally uniformly continuous. Let u be the unique viscosity solution of (4.8) with $u(\cdot, 0) = u_0(\cdot)$ bounded in \mathbb{H} . If u_0 is Lipschitz with respect to d_L in \mathbb{H} , i.e., there exists $L > 0$ such that (4.6) holds for any $p, q \in \mathbb{H}$, then for all $t \geq 0$*

$$|u(p, t) - u(q, t)| \leq Ld_L(p, q)$$

for all $p, q \in \mathbb{H}$.

Proof. Under the assumptions above, it is known [26] that for any fixed $T > 0$, there is a unique bounded continuous viscosity solution in $\mathbb{H} \times [0, T)$. We only need to show that

$$u(p, t) - u(q, t) \leq Ld_L(p, q)$$

for all $p, q \in \mathbb{H}$ and $t \in [0, T)$. The other part can be shown by a symmetric argument.

By Young's inequality applied to (4.6), we obtain

$$u_0(p) - u_0(q) \leq \frac{Ld_L(p, q)^4}{4\delta^4} + \frac{3L\delta^{\frac{4}{3}}}{4} \quad (4.10)$$

for all $\delta > 0$ and $p, q \in \mathbb{H}$. It then suffices to show that

$$u(p, t) - u(q, t) \leq \frac{Ld_L(p, q)^4}{4\delta^4} + \frac{3L\delta^{\frac{4}{3}}}{4} \quad (4.11)$$

for all $\delta > 0$ and $p, q \in \mathbb{H}$. To this end, we fix $\delta > 0$ and prove below that

$$u_L(p, t) = \inf_{q \in \mathbb{H}} \{u(q, t) + Cd_L(p, q)^4\}$$

with $C = L/4\delta^4$ is a supersolution of (4.8). Suppose there exist a bounded open set $\mathcal{O} \subset \mathbb{H} \times (0, T)$, $\phi \in C^2(\mathcal{O})$ and $(\hat{p}, \hat{t}) \in \mathcal{O}$ such that

$$(u_L - \phi)(\hat{p}, \hat{t}) < (u_L - \phi)(p, t)$$

for all $(p, t) \in \mathcal{O}$. We may also assume that $\phi(p, t) \rightarrow -\infty$ when $(p, t) \rightarrow \partial\mathcal{O}$. Then for any $\varepsilon > 0$ sufficiently small,

$$\Phi_\varepsilon(p, q, t, s) = u(q, t) + Cd_L(p, q)^4 - \phi(p, s) + \frac{(t - s)^2}{\varepsilon}$$

attains a minimum at $(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathbb{H} \times \mathbb{H} \times [0, \infty) \times [0, \infty)$. A standard argument yields $p_\varepsilon, q_\varepsilon \rightarrow \hat{p}$ and $t_\varepsilon, s_\varepsilon \rightarrow \hat{t}$ as $\varepsilon \rightarrow 0$, which, in particular, implies that $t_\varepsilon, s_\varepsilon \neq 0$. The minimum also implies that

$$\nabla_H \phi_1(p_\varepsilon) = \nabla_H \phi(p_\varepsilon, s_\varepsilon) \text{ and } \phi_t(p_\varepsilon, s_\varepsilon) = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon}, \quad (4.12)$$

where $\phi_1(p) = Cd_L(p, q_\varepsilon)^4$.

We next apply the definition of supersolutions and get

$$a + m(|\nabla_H \phi_2(q_\varepsilon)|) \geq 0, \quad (4.13)$$

where

$$a = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon} \quad \text{and} \quad \phi_2(q) = -Cd_L(p_\varepsilon, q)^4.$$

By (4.12), in order to prove that u_L is a supersolution, we only need to substitute $\nabla_H \phi_2(q_\varepsilon)$ in (4.13) with $\nabla_H \phi_1(p_\varepsilon)$. By direct calculation, we have

$$\nabla_H^p d_L(p, q)^4 = 4 \left(\delta_1(\delta_1^2 + \delta_2^2) - 4\delta_2\delta_3, \delta_2(\delta_1^2 + \delta_2^2) + 4\delta_1\delta_3 \right)$$

and

$$\nabla_H^q d_L(p, q)^4 = 4 \left(-\delta_1(\delta_1^2 + \delta_2^2) - 4\delta_2\delta_3, -\delta_2(\delta_1^2 + \delta_2^2) + 4\delta_1\delta_3 \right)$$

with $p = (x_p, y_p, z_p)$, $q = (x_q, y_q, z_q)$ and

$$\delta_1 = x_p - x_q, \quad \delta_2 = y_p - y_q, \quad \delta_3 = z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p,$$

This reveals that $\nabla_H^p d_L(p, q)^4 \neq -\nabla_H^q d_L(p, q)^4$ and therefore $\nabla_H \phi_1(p_\varepsilon) \neq \nabla_H \phi_2(q_\varepsilon)$ in general; see [12] for more details on this aspect. However, their norms stay the same, i.e., $|\nabla_H^p d_L(p, q)^4| = |\nabla_H^q d_L(p, q)^4|$, which turns out to be a key ingredient in this proof. In fact, we have

$$|\nabla_H^p d_L(p, q)^4| = |\nabla_H^q d_L(p, q)^4| = 4d_L(p, q)^2(\delta_1^2 + \delta_2^2)^{\frac{1}{2}},$$

which implies that $|\nabla_H \phi_1(p_\varepsilon)| = |\nabla_H \phi_2(q_\varepsilon)|$ and their boundedness uniformly in ε . Hence, due to (4.12), the equation (4.13) is now rewritten as

$$\phi_t(p_\varepsilon, s_\varepsilon) + m(|\nabla_H \phi(p_\varepsilon, s_\varepsilon)|) \geq 0.$$

By sending $\varepsilon \rightarrow 0$ and using the continuity of m , we conclude the verification that u_L is a supersolution. It follows that $v_L = u_L + 3L\delta^{\frac{4}{3}}/4$ is also a supersolution of (4.8). Thanks to (4.10), we have $u(p, 0) \leq v_L(p, 0)$, which implies (4.11) by Theorem 1.1. \square

5. CONVEXITY PRESERVING PROPERTIES

It is well known that the convexity preserving property holds for a large class of fully nonlinear equations in the Euclidean space; see [14]. Concerning convexity in the Heisenberg group, the notion of h-convexity (and equivalently v-convexity) turns out to be a natural extension of the Euclidean version. However, we cannot expect such convexity to be preserved in general. In fact, h-convexity is not preserved even for the first order linear equation.

Example 5.1 (Linear first order equations). We again consider the linear equation (1.9) with $h_0 = (1, 1)$ and $u(x, y, z, 0) = f(x, y, z)$ with f defined as in (2.6) for all $(x, y, z) \in \mathbb{H}$. Let $h = (1, 1, 0) \in \mathbb{H}_0$. As verified in Example 2.8, $u(\cdot, 0)$ is h-convex in \mathbb{H} . However, the unique solution

$$u(p, t) = f(p \cdot ht) = (x + t)^2(y + t)^2 + 2 \left(z + \frac{1}{2}xt - \frac{1}{2}yt \right)^2 \quad (5.1)$$

is not h -convex for any $t > 0$. In fact, the symmetrized Hessian is given by

$$(\nabla_H^2 u)^*(p, t) = \begin{pmatrix} 2(y+t)^2 + (y-t)^2 & 4(x+t)(y+t) - (x-t)(y-t) \\ 4(x+t)(y+t) - (x-t)(y-t) & 2(x+t)^2 + (x-t)^2 \end{pmatrix}.$$

It is therefore easily seen that

$$(\nabla_H^2 u)^*(t, t, 0, t) = \begin{pmatrix} 8t^2 & 16t^2 \\ 16t^2 & 8t^2 \end{pmatrix},$$

which shows that $u(\cdot, t)$ is not h -convex around the point $p = (t, t, 0) \in \mathbb{H}$ for any $t > 0$.

The loss of convexity preserving is due to the non commutativity of the Heisenberg group product. Although the h -convexity of a function is preserved under left translations, it is not necessarily preserved under right translations, as indicated in Example 5.1. We therefore consider right invariant h -convexity next.

5.1. Right invariant h -convexity preserving.

Definition 5.2 (Right invariant h -convexity). Let Ω be an open set in \mathbb{H} and $u : \Omega \rightarrow \mathbb{R}$ be an upper semicontinuous function. The function u is said to be right invariant horizontally convex or right h -convex in Ω , if for every $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$ such that $[h^{-1} \cdot p, h \cdot p] \subset \Omega$, we have

$$u(h^{-1} \cdot p) + u(h \cdot p) \geq 2u(p). \quad (5.2)$$

Theorem 5.3 (Right invariant h -convexity preserving). *Suppose that the assumptions (A1), (A2) and (A3) hold. Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique viscosity solution of (1.1)–(1.2) satisfying the growth condition (G). Assume in addition that f is right invariant concave in $\mathbb{H} \times \mathbb{R}^2$, i.e.,*

$$f(h^{-1} \cdot p, \xi) + f(h \cdot p, \eta) \leq 2f\left(p, \frac{1}{2}(\xi + \eta)\right) \quad (5.3)$$

for all $p \in \mathbb{H}$, $h \in \mathbb{H}_0$ and $\xi, \eta \in \mathbb{R}^2$. If u_0 is right invariant h -convex in \mathbb{H} , then so is $u(\cdot, t)$ for all $t \geq 0$.

Proof of Theorem 5.3. By definition, we aim to show that

$$u(h^{-1} \cdot p, t) + u(h \cdot p, t) \geq 2u(p, t)$$

for any $p \in \mathbb{H}$, $h \in \mathbb{H}_0$, $t \geq 0$. We assume by contradiction that there exist $(p_0, h_0, t_0) \in \mathbb{H} \times \mathbb{H}_0 \times [0, \infty)$ such that

$$u(h_0^{-1} \cdot p_0, t_0) + u(h_0 \cdot p_0, t_0) < 2u(p_0, t_0).$$

Then there exists a positive maximizer $(\hat{p}, \hat{h}, \hat{t}) \in \mathbb{H} \times \mathbb{H}_0 \times [0, T)$ of

$$2u(p, t) - u(h^{-1} \cdot p, t) - u(h \cdot p, t) - 3\sigma g(p, t) - \frac{\sigma}{m - |h|_G^4} - \frac{\sigma}{T - t}$$

with some constants $m > |\hat{h}|_G^4$, $T > \hat{t}$ and $\sigma > 0$ small. Here $g(p, t) = e^{\alpha t + \beta \langle p \rangle}$ with $\alpha > 0$ to be determined later and any fixed $\beta > k$. We next consider

$$\begin{aligned} \Phi(p, q, r, h, t, s, \tau) = & 2u(r, \tau) - u(h^{-1} \cdot p, t) - u(h \cdot q, s) - \sigma \Psi_\varepsilon(p, q, r, t, s, \tau) \\ & - \psi_\varepsilon(t, s, \tau) - \frac{\sigma}{m - |h|_G^4} - \frac{\sigma}{T - \tau}, \end{aligned}$$

where

$$\begin{aligned} \psi_\varepsilon(t, s, \tau) &= \frac{(t-s)^2}{\varepsilon} + \frac{(t-\tau)^2}{\varepsilon} + \frac{(s-\tau)^2}{\varepsilon}, \\ \Psi_\varepsilon(p, q, r, t, s, \tau) &= \phi_\varepsilon(p, q, r) + K(p, q, r, t, s, \tau) \end{aligned}$$

with

$$\phi_\varepsilon(p, q, r) = \frac{|p \cdot r^{-1}|^4}{\varepsilon} + \frac{|q \cdot r^{-1}|^4}{\varepsilon}$$

and

$$K(p, q, r, t, s, \tau) = g(r, \tau) + g(p, t) + g(q, s).$$

It follows that Φ has a maximizer $(p_\varepsilon, q_\varepsilon, r_\varepsilon, h_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon)$. As before, we denote

$$p_\varepsilon = (x_{p_\varepsilon}, y_{p_\varepsilon}, z_{p_\varepsilon}), \quad q_\varepsilon = (x_{q_\varepsilon}, y_{q_\varepsilon}, z_{q_\varepsilon}), \quad r_\varepsilon = (x_{r_\varepsilon}, y_{r_\varepsilon}, z_{r_\varepsilon}).$$

Due to the penalization at space infinity, we have $\phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon)$ bounded from above uniformly in ε . By a standard argument, we can show that there exists $\bar{p} \in \mathbb{H}$, $\bar{h} \in \mathbb{H}_0$ and $\bar{t} \in [0, T]$ such that, up to a subsequence,

$$p_\varepsilon, q_\varepsilon, r_\varepsilon \rightarrow \bar{p}, \quad h_\varepsilon \rightarrow \bar{h}, \quad t_\varepsilon, s_\varepsilon, \tau_\varepsilon \rightarrow \bar{t} \quad (5.4)$$

and

$$\phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon) \rightarrow 0 \quad (5.5)$$

as $\varepsilon \rightarrow 0$.

Since u_0 is right invariant h-convex, we have $\bar{t} > 0$ and therefore $t_\varepsilon, s_\varepsilon, \tau_\varepsilon > 0$ when ε is sufficiently small.

Denote $u_-(p, t) = u(h_\varepsilon^{-1} \cdot p, t)$ and $u_+(p, t) = u(h_\varepsilon \cdot p, t)$. We now apply the Crandall-Ishii lemma in the Heisenberg group and get, for any $\lambda \in (0, 1)$,

$$\begin{aligned} (a_1, \zeta_1, X_1) &\in \bar{J}^{2,-} u_-(p_\varepsilon, t_\varepsilon), \quad (a_2, \zeta_2, X_2) \in \bar{J}^{2,-} u_+(q_\varepsilon, s_\varepsilon), \\ (a_3, \zeta_3, X_3) &\in \bar{J}^{2,+} u(r_\varepsilon, \tau_\varepsilon) \end{aligned}$$

such that

$$2a_3 - a_1 - a_2 = \frac{\sigma}{(T - \tau_\varepsilon)^2} + \sigma \alpha K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon), \quad (5.6)$$

the horizontal projections of ζ_i can be expressed as $\xi_i + \eta_i$ ($i = 1, 2, 3$) with

$$\begin{aligned} -\xi_1 &= \sigma \nabla_H^p \phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon), \quad -\xi_2 = \sigma \nabla_H^q \phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon), \quad 2\xi_3 = \sigma \nabla_H^r \phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon), \\ -\eta_1 &= \sigma \nabla_H g(p_\varepsilon, t_\varepsilon), \quad -\eta_2 = \sigma \nabla_H g(q_\varepsilon, s_\varepsilon), \quad 2\eta_3 = \sigma \nabla_H g(r_\varepsilon, \tau_\varepsilon), \end{aligned}$$

and

$$\langle (2X_3 - X_1 - X_2)w, w \rangle \leq \langle (\sigma M + \lambda \sigma^2 M^2)w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle, \quad (5.7)$$

for all $w \in \mathbb{R}^2$, where $M = \nabla^2 \Psi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon, h_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon)$ is a 9×9 symmetric matrix, $w_{p_\varepsilon}, w_{q_\varepsilon}$ are respectively taken as in (3.8) and (3.9), and

$$w_{r_\varepsilon} = \left(w_1, w_2, \frac{1}{2}w_2x_{r_\varepsilon} - \frac{1}{2}w_1y_{r_\varepsilon} \right).$$

By calculation, it is easily seen that

$$2\xi_3 = \xi_1 + \xi_2. \quad (5.8)$$

We next set

$$M_1 = \nabla^2 \phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon)$$

and

$$M_2 = \nabla^2 K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon)$$

Then $M = M_1 + M_2$. To investigate the right hand side of (5.7), we first give estimates the terms involving M_1 , which is a variant of (3.10) for three space variables. Note that

$$M_1 = M'_1 + M''_1,$$

where

$$M'_1 = \frac{1}{\varepsilon} \nabla^2 |p \cdot r^{-1}|^4, \quad M''_1 = \frac{1}{\varepsilon} \nabla^2 |q \cdot r^{-1}|^4.$$

By direct calculations, we get

$$\begin{aligned} \langle M'_1 w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle &= 0, \\ \langle M''_1 w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle M_1'^2 w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle &= \frac{512}{\varepsilon} m_1^2 |w|^2, \\ \langle M_1''^2 w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle &= \frac{512}{\varepsilon} m_2^2 |w|^2, \\ \langle M'_1 M''_1 w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle &= \frac{256}{\varepsilon} m_1 m_2 |w|^2, \end{aligned}$$

where

$$m_1 = z_{p_\varepsilon} - z_{r_\varepsilon} + \frac{1}{2}y_{p_\varepsilon}x_{r_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{r_\varepsilon}, \quad m_2 = z_{q_\varepsilon} - z_{r_\varepsilon} + \frac{1}{2}y_{q_\varepsilon}x_{r_\varepsilon} - \frac{1}{2}x_{q_\varepsilon}y_{r_\varepsilon}.$$

It follows that

$$\langle (M_1 + \lambda M_1^2) w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle \leq \frac{512\lambda}{\varepsilon} (m_1^2 + m_2^2 + m_1 m_2) |w|^2, \quad (5.9)$$

which implies that

$$\langle (M_1 + \lambda M_1^2) w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle \leq \frac{C}{\varepsilon} |w|^2 (|p_\varepsilon \cdot r_\varepsilon^{-1}|^4 + |q_\varepsilon \cdot r_\varepsilon^{-1}|^4) \quad (5.10)$$

for some $C > 0$ independent of ε and λ . On the other hand, with the help of computer algebra system, we obtain estimates similar to (3.11), (3.12) and (3.13). In fact, we get a constant C_β such that, when $\lambda > 0$ is small enough (depending on ε),

$$\langle M_2(w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}) \rangle \leq C_\beta |w|^2 K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon) \quad (5.11)$$

$$\begin{aligned} & \langle \lambda(M_1 M_2 + M_2 M_1 + M_2^2)(w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}) \rangle \\ & \leq 2C_\beta |w|^2 K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon) \end{aligned} \quad (5.12)$$

for some constant $\mu > 0$ independent of ε, β and σ and satisfying (3.3). As remarked in the proof of Theorem 1.1, we obtain the constant C_β thanks to the boundedness of $\nabla_H \langle p \rangle$ and $\nabla_H^2 \langle p \rangle$ in \mathbb{H} .

Combining (5.7), (5.10) and (5.12), we have

$$\begin{aligned} \langle (2X_3 - X_1 - X_2)w, w \rangle & \leq \frac{C\sigma}{\varepsilon} \|w\|^2 (|p_\varepsilon \cdot r_\varepsilon^{-1}|^4 + |q_\varepsilon \cdot r_\varepsilon^{-1}|^4) \\ & \quad + 2\sigma \|w\|^2 C_\beta K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon). \end{aligned} \quad (5.13)$$

when $\lambda > 0$ and $\sigma > 0$ are sufficiently small.

Since the horizontal derivatives are left translation invariant, the functions u_- and u_+ are respectively solutions of

$$(u_-)_t - \text{tr}(A \nabla_H^2 u_-) + f(h_\varepsilon^{-1} \cdot p, \nabla_H u_-) = 0 \quad \text{in } \mathbb{H} \times (0, \infty)$$

and

$$(u_+)_t - \text{tr}(A \nabla_H^2 u_+) + f(h_\varepsilon \cdot p, \nabla_H u_+) = 0 \quad \text{in } \mathbb{H} \times (0, \infty).$$

Applying the definition of viscosity subsolutions and supersolution, we have

$$a_1 - \text{tr}(AX_1) + f(h_\varepsilon^{-1} \cdot p_\varepsilon, \xi_1 + \eta_1) \geq 0, \quad (5.14)$$

$$a_2 - \text{tr}(AX_2) + f(h_\varepsilon \cdot q_\varepsilon, \xi_2 + \eta_2) \geq 0, \quad (5.15)$$

$$a_3 - \text{tr}(AX_3) + f(r_\varepsilon, \xi_3 + \eta_3) \leq 0. \quad (5.16)$$

Subtracting (5.14) and (5.15) from twice (5.16), we get

$$2a_3 - a_1 - a_2 \leq \text{tr} A(2X_3 - X_1 - X_2) + E, \quad (5.17)$$

where

$$E = f(h_\varepsilon^{-1} \cdot p_\varepsilon, \xi_1 + \eta_1) + f(h_\varepsilon \cdot q_\varepsilon, \xi_2 + \eta_2) - 2f(r_\varepsilon, \xi_3 + \eta_3)$$

It follows from the concavity assumption (5.3), the relation (5.8) and (A1)-(A2) that

$$\begin{aligned} E & \leq f(h_\varepsilon^{-1} \cdot p_\varepsilon, \xi_1 + \eta_1) - f(h_\varepsilon^{-1} \cdot r_\varepsilon, \xi_1 + \eta_1) + f(h_\varepsilon \cdot q_\varepsilon, \xi_2 + \eta_2) - f(h_\varepsilon \cdot r_\varepsilon, \xi_2 + \eta_2) \\ & \quad + f(h_\varepsilon^{-1} \cdot r_\varepsilon, \xi_1 + \eta_1) + f(h_\varepsilon \cdot r_\varepsilon, \xi_2 + \eta_2) - 2f(r_\varepsilon, \frac{1}{2}(\xi_1 + \xi_2 + \eta_1 + \eta_2)) \\ & \quad + 2f(r_\varepsilon, \frac{1}{2}(\xi_1 + \xi_2 + \eta_1 + \eta_2)) - 2f(r_\varepsilon, \xi_3 + \eta_3) \\ & \leq L_R(|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot p_\varepsilon^{-1} \cdot h| + |h_\varepsilon^{-1} \cdot r_\varepsilon \cdot q_\varepsilon^{-1} \cdot h_\varepsilon|) + L_f |\eta_1 + \eta_2 - 2\eta_3| \end{aligned} \quad (5.18)$$

with $R = (|\overline{p}| + 1)$ and $\varepsilon > 0$ small. Also, by (3.3), we have

$$|\eta_1 + \eta_2 - 2\eta_3| \leq 2(|\eta_1| + |\eta_2| + |\eta_3|) = 2\sigma\beta\mu K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon).$$

In view of (5.13), (5.17) and (5.18), we then obtain

$$\begin{aligned} & 2a_3 - a_1 - a_2 \\ & \leq \frac{C\sigma}{\varepsilon}(|p_\varepsilon \cdot r_\varepsilon^{-1}|^4 + |q_\varepsilon \cdot r_\varepsilon^{-1}|^4) + L_R|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot p_\varepsilon^{-1} \cdot h| + L_R|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot q_\varepsilon^{-1} \cdot h_\varepsilon| \quad (5.19) \\ & \quad + 2\sigma(C_\beta\|A\| + L_f\beta\mu)K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon). \end{aligned}$$

In view of (5.4) and (5.5), we can take $\varepsilon > 0$ small such that

$$\frac{C\sigma}{\varepsilon}(|p_\varepsilon \cdot r_\varepsilon^{-1}|^4 + |q_\varepsilon \cdot r_\varepsilon^{-1}|^4) + L_R|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot p_\varepsilon^{-1} \cdot h| + L_R|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot q_\varepsilon^{-1} \cdot h_\varepsilon| < \frac{\sigma}{T^2},$$

which, by (5.19), implies

$$2a_3 - a_1 - a_2 \leq \frac{\sigma}{T^2} + 2\sigma(C_\beta\|A\| + L_f\beta\mu)K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon).$$

It clearly contradicts (5.6) when α is chosen to satisfy

$$\alpha > 2\|A\|C_\beta + 2L_f\beta\mu.$$

□

Remark 5.4. The concavity assumption (5.3) on the operator f is stronger than the assumptions of the convexity results in the Euclidean space as shown in [14, 17]. In particular, the concavity of $\xi \mapsto f(p, \xi)$ is not needed in the Euclidean case. We here need this assumption, since there are no expressions of h-convexity in \mathbb{H} corresponding to the following one for the Euclidean convexity

$$u(\xi) + u(\eta) \geq 2u\left(\frac{\xi + \eta}{2}\right)$$

for all $\xi, \eta \in \mathbb{R}^n$. It is not clear to us whether the assumption (5.3) can be weakened.

Example 5.5. Let us revisit Example 5.1. Since the equation (1.9) and the solution (5.1) satisfy all of the assumptions in Theorem 5.3, the right invariant h-convexity of the solution is preserved, though the h-convexity is not. Indeed, if $u(p, t)$ is given by (5.1), then by direct calculation we obtain, for all $p = (x, y, z)$, $h = (h_1, h_2, 0)$ and $t \geq 0$,

$$\begin{aligned} & u(h \cdot p, t) + u(h^{-1} \cdot p, t) \\ & = (x + h_1 + t)^2(y + h_2 + t)^2 + 2\left(z + \frac{1}{2}h_1y - \frac{1}{2}h_2x + \frac{1}{2}(x + h_1)t - \frac{1}{2}(y + h_2)t\right)^2 \\ & = 2(x + t)^2(y + t)^2 + 4\left(z + \frac{1}{2}xt - \frac{1}{2}yt\right)^2 + (h_1(y + t) - h_2(x + t))^2 + 2h_1^2(y + t)^2 \\ & \quad + 2h_2^2(x + t)^2 + 8(x + t)(y + t)h_1h_2 + 2h_1^2h_2^2 \\ & = 2(x + t)^2(y + t)^2 + 4\left(z + \frac{1}{2}xt - \frac{1}{2}yt\right)^2 + 3(h_1(y + t) + h_2(y + t))^2 + 2h_1^2h_2^2 \\ & \geq 2(x + t)^2(y + t)^2 + 4\left(z + \frac{1}{2}xt - \frac{1}{2}yt\right)^2 = 2u(p, t). \end{aligned}$$

5.2. Left invariant h-convexity preserving. We next discuss some special cases, where h-convexity and right invariant h-convexity are equivalent.

Proposition 5.6 (Evenness). *Let u be an even or vertically even function on \mathbb{H} . Then u is h-convex in \mathbb{H} if and only if u is right invariant h-convex in \mathbb{H} .*

Proof. By definition, u is h-convex if u satisfies (2.4) for any $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$. Since u is even, it is easily seen that (2.4) holds if and only if

$$u(h \cdot \bar{p}) + u(h^{-1} \cdot \bar{p}) \geq u(\bar{p}),$$

where \bar{p} is given as in (2.3), or

$$u(h \cdot p^{-1}) + u(h^{-1} \cdot p^{-1}) \geq u(p^{-1})$$

for all $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$, which is equivalent to saying

$$u(h \cdot p) + u(h^{-1} \cdot p) \geq u(p) \text{ for all } p \in \mathbb{H} \text{ and } h \in \mathbb{H}_0.$$

□

Another sufficient condition for equivalence between the h-convexity and the left h-convexity of a function u on \mathbb{H} is that u has a separate structure; namely,

$$u(x, y, z) = f(x, y) + g(z) \tag{5.20}$$

for any $(x, y, z) \in \mathbb{H}$.

Proposition 5.7 (Separability). *Let u be a function on \mathbb{H} with a separate structure as in (5.20). Then u is h-convex in \mathbb{H} if and only if u is right invariant h-convex in \mathbb{H} .*

Proof. Suppose u can be written as in (5.20). Setting $p = (x, y, z)$ and $h = (h_1, h_2)$, we then have

$$\begin{aligned} u(p \cdot h) &= f(x + h_1, y + h_2) + g(z + \frac{1}{2}xh_2 - \frac{1}{2}yh_1); \\ u(p \cdot h^{-1}) &= f(x - h_1, y - h_2) + g(z - \frac{1}{2}xh_2 + \frac{1}{2}yh_1); \\ u(h \cdot p) &= f(x + h_1, y + h_2) + g(z + \frac{1}{2}yh_1 - \frac{1}{2}xh_2); \\ u(h^{-1} \cdot p) &= f(x - h_1, y - h_2) + g(z - \frac{1}{2}yh_2 + \frac{1}{2}xh_2). \end{aligned}$$

It is easily seen that in this case

$$u(p \cdot h^{-1}) + u(p \cdot h) = u(h^{-1} \cdot p) + u(h \cdot p),$$

which immediately yields the equivalence of (2.4) and (5.2) in \mathbb{H} . □

The following result on preserving of the h-convexity itself is an immediate consequence of Theorem 5.3, Propositions 5.6 and 5.7.

Corollary 5.8 (H-convexity preserving under evenness or separability). *Assume that f satisfies (A1)–(A3) and the concavity condition (5.3) for all $p \in \mathbb{H}$, $h \in \mathbb{H}_0$ and $\xi, \eta \in \mathbb{R}^2$. Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique viscosity solution of (1.1)–(1.2) satisfying the growth condition (G). Assume in addition that for any $t \geq 0$, $u(\cdot, t)$ either is an even or vertically even function or has a separable structure as in (5.20). If u_0 is h-convex in \mathbb{H} , then so is $u(\cdot, t)$ in \mathbb{H} for all $t \geq 0$.*

5.3. More examples. In this section, we provide more examples, where the h-convexity is preserved.

Example 5.9. Let $u_0(x, y, z) = (x^2 + y^2)^2 - 8z^2$. It is not difficult to see that u_0 is an h-convex function in \mathbb{H} . Consider the heat equation

$$u_t - \Delta_H u = 0 \quad \text{in } \mathbb{H} \times (0, \infty) \quad (5.21)$$

with $u(\cdot, 0) = u_0$ in \mathbb{H} , where Δ_H denotes the horizontal Laplacian operator in the Heisenberg group, i.e., $\Delta_H u = \text{tr}(\nabla_H^2 u)^*$. The unique solution of (5.21) in this case is

$$u(x, y, z, t) = (x^2 + y^2)^2 - 8z^2 + 12(x^2 + y^2)t + 24t^2 \quad (5.22)$$

for all $(x, y, z) \in \mathbb{H}$ and $t \geq 0$ and it actually preserves the h-convexity of the initial value u_0 .

Example 5.10. The solution as in (5.22) looks special, since it can be written as the sum of a function of x, y, t and a function of z . A more complicated solution of the heat equation (5.21) is

$$u(x, y, z, t) = (x^2 + y^2)z^2 + \frac{1}{24}(x^2 + y^2)^3 + (4z^2 + 2(x^2 + y^2)^2)t + 17(x^2 + y^2)t^2 + \frac{68}{3}t^3 \quad (5.23)$$

which contains mixed terms of x, y and z . By direct calculation, one can also show that $u(\cdot, t)$ satisfies (2.5) in \mathbb{H} in the classical sense for everywhere $t \geq 0$.

Example 5.11. We recall another example in [12] for the level-set mean curvature flow equation in \mathbb{H} . The equation is of the form

$$u_t - |\nabla_H u| \text{div}_H \left(\frac{\nabla_H u}{|\nabla_H u|} \right) = 0 \quad \text{in } \mathbb{H} \times (0, \infty), \quad (5.24)$$

where div_H stands for the horizontal divergence operator in the Heisenberg group. An explicit solution is

$$u(x, y, z, t) = (x^2 + y^2)^2 + 16z^2 + 12(x^2 + y^2)t + 12t^2.$$

This is also an example of h-convexity preserving but unfortunately is not covered by our current results.

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